Int. J. Solids Structures, 1970, Vol. 6, pp. 1011 to 1019. Pergamon Press. Printed in Great Britain

# COLUMN INSTABILITIES CAUSED BY CYCLIC LOADING

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**Abstract**—The consequences of the existence of second order plastic strains as applied to the problem of cyclic loading of columns are investigated. The possibility of columns safely designed for a small number of loading cycles becoming unstable for long term cycling is shown to exist. Two examples are considered. The first, cycling between fixed end displacements and the second cycling between fixed loads. In both examples buckling can result although in the first example a proper interpretation of the constraints must be made if buckling is to be possible. A simple plasticity model exhibiting second order effects is developed for use in the second example.

## 1. INTRODUCTION

THE PHENOMENON of the accumulation of permanent plastic strain in metal specimens subjected to steady state stress cycling has been experimentally investigated for homogeneous stress conditions [1–5]. This phenomenon, which has been termed "cyclic creep" is a dramatic departure from the material behavior one might expect based on the knowledge of a single hysteresis loop. Briefly, a uniaxial specimen when stress cycled about some mean stress will experience a strain accumulation in the direction of the mean stress. Strain accumulation seems to be experienced at small stress amplitude and small mean stress although the rate of accumulation increases with amplitude and mean stress as is to be expected. There is as yet no evidence that the phenomenon vanishes (or approaches zero) when the number of cycles becomes large. Hence, one can assume that the rate of accumulation reaches a non zero limit. The present study investigates the possibility of column buckling under this assumption.

Apparently there have been no analytical investigations of the consequences of cyclic strain accumulation in problems of structural members subjected to non homogeneous stress states. The possibilities are intriguing. One can show that the stresses in a beam subjected to a constant axial load and a cyclic moment may become unbounded (assuming a plastic strain hardening material and neglecting viscoelastic effects). In other applications the strain accumulation acts very much like a creep phenomenon. Two such applications are discussed here.

After a brief summary of experimental observations we examine a straight column undergoing a cyclic strain about some constant mean tensile strain. In general, it is shown that the compressive load will during some later strain cycle exceed the Euler buckling load. Then, a simple plasticity model is introduced which predicts a second order strain accumulation and is employed in the analysis of a sandwich column with an initial curvature subject to cyclic loading about a constant mean compressive load. In general, the column curvature becomes unbounded.

## 2. TWO EXPERIMENTAL OBSERVATIONS

Consider a prismatic bar subjected to longitudinal strain cycling of constant amplitude about a fixed mean strain. The mean stress over each cycle becomes small with corresponding changes in the maximum and minimum stresses as the number of cycles becomes large [6, 7], Fig. 1. (For our purpose it is not necessary to assume that the mean stress approaches zero in the limit but merely that it becomes small with the stress amplitude remaining approximately constant.)

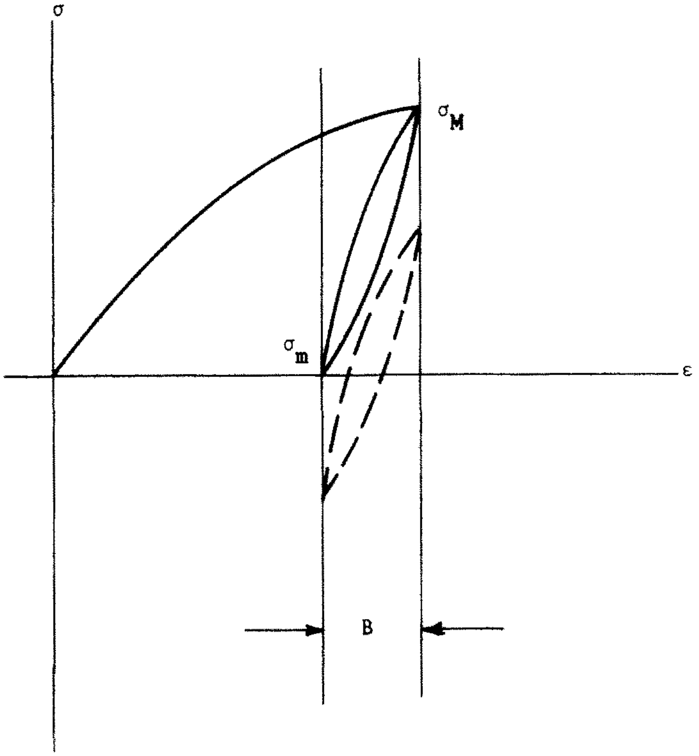


FIG. 1. Relaxation of mean stress for steady state strain cycling. Maximum and minimum stresses for a given cycle are denoted by $\sigma_M$ and $\sigma_m$, respectively.

If now a prismatic bar is subjected to cyclic axial loading with a fixed tensile mean stress and a fixed stress amplitude, an extensional increment of permanent strain will accumulate with each cycle in the direction of the mean stress [1, 2]. The rate of accumulation increases with an increase in mean stress and stress amplitude (see Fig. 2 for a schematic diagram). For our purposes it is convenient to assume that this accumulation achieves steady state conditions. There is no experimental evidence to suggest the contrary. If, in fact, it is later shown that steady state is not achieved the result of the second example presented

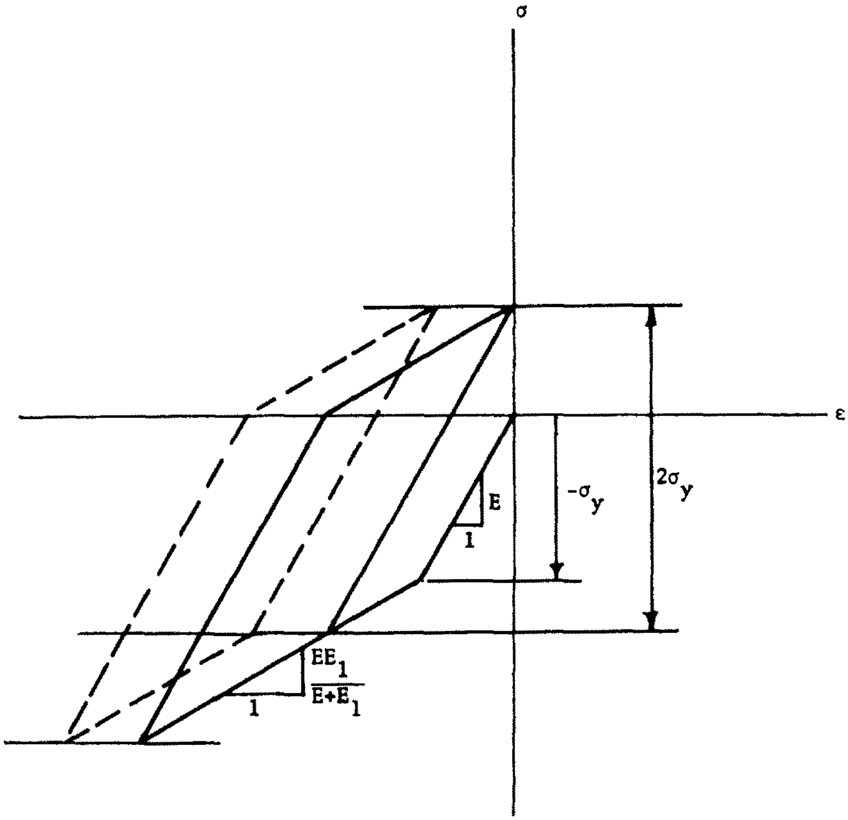


FIG. 2. Schematic hysteresis loops exhibiting accumulation effect. Fixed mean stress $\sigma_0$ is negative for example shown. Solid line loop neglects second order effects and exhibits no accumulation.

here will be affected in that the column displacement will become "large" instead of unbounded.

## 3. BIFURCATION OF A COLUMN UNDERGOING CYCLIC STRAIN

The results of this first example follow immediately from the phenomenon of stress relaxation due to strain cycling. A simple analysis is possible because we may treat column buckling as an eigenvalue problem whereby if the Euler load is ever attained in a perfectly straight column bifurcation results. This approach offers the simplification of homogeneous stress fields; the qualitative picture of stress relaxation observed for homogeneous stress experiments provides the results.

Consider a straight simply supported column loaded along its longitudinal centroidal axis. Subject the column to a load history which produces a cyclic strain history†.

$$\varepsilon = \varepsilon_0 + \varepsilon_1 \sin t, \qquad \varepsilon_0 > 0, \qquad \varepsilon_1 > 0. \tag{1}$$

† The example is somewhat artificial in that we require a controlled longitudinal strain history where this history is produced by controlling the end loads not by controlling the end displacements. We require the situation where the column is not restrained from buckling which would not be the case with controlled end displacements.

As strain cycling proceeds the mean stress $\sigma_0$ relaxes and the minimum $\sigma_m$ and maximum $\sigma_M$ stresses occurring in each cycle correspondingly decrease. In fact, $\sigma_m$ and $\sigma_M$ asymptotically approach values which are independent of $\varepsilon_0$ and depend only upon $\varepsilon_1$ in (1). If during any cycle the compressive stress reaches the Euler buckling load the column will collapse. This possibility of collapse does not depend upon the existence of asymptotic properties which have been implied here, i.e. $\lim_{t\to\infty} = \sigma_0 = 0$; the result depends only upon the experimentally observed fact that $\sigma_0$ does relax towards zero over a large number of cycles.

It is interesting that if $\varepsilon_0 > \varepsilon_1$ the column will never experience compressure strain. Nevertheless compressive stresses will be produced and if $\varepsilon_1$ is sufficiently large the column will buckle.

In the case of an initially curved column this simplified analysis is not applicable. Nevertheless, the qualitative picture remains. If such a column is cycled between fixed end displacements the mean end load will relax and the minimum load over a cycle will decrease.

## 4. A SIMPLE ONE-DIMENSIONAL PLASTICITY MODEL

In this section we are interested in developing, for analytical use, a simple one-dimensional plasticity model which exhibits a second order effect. We consider the one-dimensional specialization of the addition of a second order effect to Ziegler's modification [8] of Pragers' kinematic model [9]. Any such kinematic model has the disadvantage that a finite elastic region is contained within the yield locus. That is, plastic strain is produced only when the stress point moves the rigid yield locus whereas purely elastic action results from motion of the stress point inside the locus. In reality, experiments suggest that there is no such thing as the conventional yield limit; second order plastic strains are produced by stress action in the nominal elastic region (inside the yield locus of the kinematic model). The kinematic model therefore neglects some of the second order strains.

In one dimension the model consists of the addition of a small cubic term to the familiar piecewise linear strain hardening law. Upon first application of a monatonically varying load the stress strain relation is

$$\varepsilon_1 = \frac{\sigma}{E} + \frac{\beta_1}{E_1}(\sigma - \sigma_y) + \frac{\beta_1}{E_2^3}(\sigma^3 - \sigma_y^3); \qquad \beta_1 = \begin{Bmatrix} 0, |\sigma| < \sigma_y \\ \\ 1, |\sigma| > \sigma_y \end{Bmatrix}. \tag{2a}$$

The constants, Fig. 2, are: $E$—elastic Young's modulus, $E_1$—linearized modulus, $E_2 \gg E$—second order modulus, $\sigma_y > 0$—initial yield stress.

For our purposes it is sufficient to consider stress histories where the continuous stress–time variation between any two consecutive relative maximum or minimum is monotonic. We refer to the stress history during such a time interval as a half cycle. Two half cycles comprise a sort of stress cycle. Furthermore we consider only the case where plastic action takes plastic during each half cycle. Thus the stress point is in contact with the yield locus at the beginning and at the end of each half cycle with plastic action taking place at the end of the half cycle.

Let the index $n \geq 1$ refer to the $n$th half cycle and let $\sigma_n(\varepsilon_n)$ be the stress (strain) at the end of the $n$th half cycle. For the $(n+1)$th half cycle

$$\varepsilon-\varepsilon_n = \frac{\sigma-\sigma_n}{E}+\frac{\beta_2}{E_1}[\sigma-(\sigma_n+2\beta\sigma_y)]+\frac{\beta_2}{E_2^3}[\sigma^3-(\sigma_n+2\beta\sigma_y)^3]$$

$$\beta_2 = \begin{Bmatrix} 1 \text{ if } |\sigma-\sigma_n| \geq 2\sigma_y \\ 0 \text{ otherwise} \end{Bmatrix} \tag{2b}$$

$$\beta = \begin{Bmatrix} 1 \text{ if } \sigma-\sigma_n > 0 \\ -1 \text{ if } \sigma-\sigma_n < 0 \end{Bmatrix}.$$

Equations (2) are in qualitative agreement with the second order strain accumulation phenomenon. Under cyclic stress the accumulation increases with an increase in mean stress (vanishing for zero mean stress) and with an increase in stress amplitude. A quadratic instead of a cubic term could have been employed but the cubic form was used since an odd function is more natural. Experimental evidence is not sufficient to suggest the exact dependence of the second order term upon mean stress and stress amplitude.

For stress cycling if the mean stress vanishes or $E_2 = \infty$ the model predicts no strain accumulation and the hysteresis loop is a steady state piecewise-linear loop shown by the solid line in Fig. 2. If the mean stress does not vanish and $E_2 \neq 0$ there is a strain accumulation and the hysteresis loop is as shown by the darked lines of Fig. 2.

The cyclic accumulation is easily determined. For example, if the stress varies between $-S$ and zero then the strain accumulation per cycle from (2b) is

$$\Delta\varepsilon = \frac{-6}{E_2^3}S\sigma_y(S-2\sigma_y). \tag{2c}$$

The strain accumulation per cycle, however, is negligible compared even to the width of the hysteresis loop.

Certainly (2) is not unique in predicting strain accumulation nor, as previously remarked, is it in qualitative agreement with all the aspects of the second order plastic phenomenon. For example the model does not predict cyclic hardening and softening [10]. Such generalizations could be included by various means [11]. For our present use the simplest model in qualitative agreement with the accumulation phenomenon is the best model.

## 5. COLLAPSE OF COLUMNS

Consider a simply supported column which initially has a small permanent lateral deflection

$$u_I = \delta\cos\left(\frac{\pi x}{L}\right) \tag{3}$$

where $x$ is a measure of distance along the longitudinal axis of the beam, $L$ is the length of the beam, and $\delta$ is the initial deflection at the center of the beam ($x = 0$). Assume that

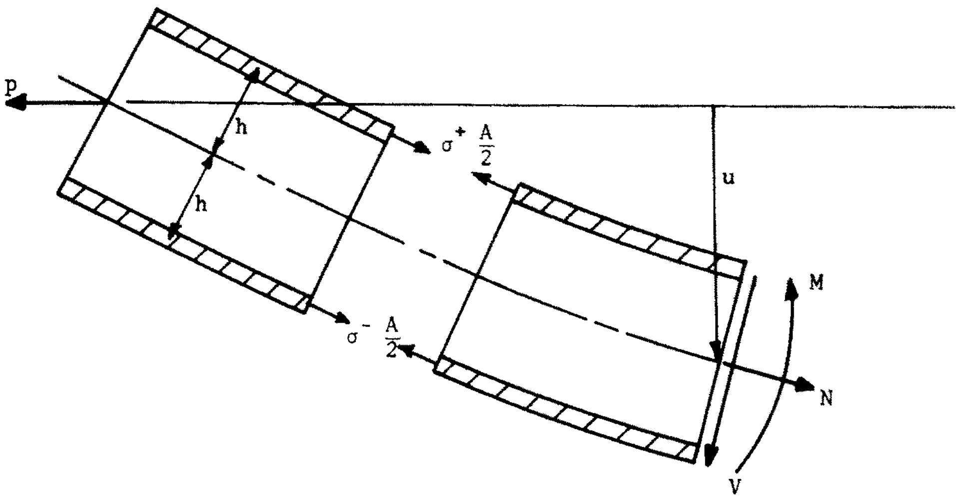


FIG. 3. Axial, shear and moment resultants acting on a typical beam element.

the column is subjected to steady state cyclic loading where the load $p$ varies monotonically between the maximum load, $p = 0$, and the minimum load, $p = -P$ $(P > 0)$.

Since the beam is initially curved the cyclic loading will produce different values of mean stress in beam fibers on opposite sides of the neutral axis and, thus, different amounts of strain accumulation in these fibers during plastic flow. Therefore, if plane sections of the beam are to remain plane the curvature of the beam must change from cycle to cycle. The change of curvature should accelerate and eventually lead to the collapse of the column. Indeed, this possibility can be shown to exist for a two-element sandwich column made of a material which responds according to (2).

The state of stress in each element is assumed to be one-dimensional, consisting only of the axial stress $\sigma$ associated with the axial strain $\varepsilon$. The stress (strain) in the top fiber is denoted by $\sigma^+(\varepsilon^+)$ while that in the bottom fiber is denoted by $\sigma^-(\varepsilon^-)$. The cross-sectional area of each element is $A/2$. For small deformation the equations of equilibrium corresponding to the element of the beam shown in Fig. 3, are

$$N = +p; \qquad V = N\frac{\mathrm{d}u}{\mathrm{d}x}; \qquad M = -pu \tag{4a}$$

where $N$, $V$ and $M$ are the axial, shear, and moment resultants; $u$ is the deflection of the centerline of the beam. The expressions for $N$ and $M$ in terms of the stresses are

$$N = \frac{A}{2}(\sigma^+ + \sigma^-); \qquad M = \frac{hA}{2}(\sigma^+ - \sigma^-) \tag{4b}$$

Relations (4a) and (4b) combine to give

$$\sigma^+ = \frac{N}{A}\left(\frac{u}{h}+1\right); \qquad \sigma^- = \frac{N}{A}\left(1-\frac{u}{h}\right). \tag{4c}$$

Plane cross-sections ($x$ = constant) before deformation are assumed to remain plane during deformation. Thus, the strain–displacement relation for two-element sandwich beam having an initial curvature is

$$\frac{1}{2h}(\varepsilon^{+}-\varepsilon^{-}) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(u-u_I) \tag{5}$$

where $h$ is half the thickness of the beam.

The plastic deformation of even this simple beam can be quite complicated. During a load cycle either or both elements may develop plastic strains at every cross-section, or plastic strains may occur at some cross-sections while only elastic strains occur at others. The boundary between the elastic and plastic regions in general varies with the load.

We consider the case where plastic strains are developed throughout the beams at the end of each half cycle of load. Moreover, the loading action is restricted to the case where

$$\Delta\sigma^{+}, \qquad \Delta\sigma^{-} = \begin{Bmatrix} < 0 \text{ if } \Delta p < 0 \\ > 0 \text{ if } \Delta p > 0 \end{Bmatrix}. \tag{6}$$

That these conditions are in fact realized will be verified *a posteriori* for a range of the loading parameters.

With the assumptions (6), the relations (2a), (4c) and (5) combine to give the following governing equation for displacements at the end of an initial half cycle where $\Delta p < 0$

$$-2h\frac{\mathrm{d}^2}{\mathrm{d}x^2}(u-u_I) = \frac{2P}{A}\left(\frac{1}{E}+\frac{1}{E_1}\right)\frac{u}{h}+\left(\frac{P}{AE_2}\right)^3\left[\left(1+\frac{u}{h}\right)^3-\left(1-\frac{u}{h}\right)^3\right] \tag{7a}$$

At the end of subsequent half cycles where $\Delta p > 0$, the relations (2b), (4c) and (5) combine to give

$$2h\frac{\mathrm{d}}{\mathrm{d}x^2}(u-u_n) = \frac{2P}{A}\left(\frac{1}{E}+\frac{1}{E_1}\right)\frac{u_n}{h}+\left(\frac{P}{AE_2}\right)^3\left[\left(1+\frac{u_n}{h}-\frac{2\sigma_y A}{P}\right)^3-\left(1-\frac{u_n}{h}-\frac{2\sigma_y A}{P}\right)^3\right]. \tag{7b}$$

The displacement at the end of the last stress action is $u_n$. The governing equation at the end of a subsequent half cycle where $\Delta p > 0$ is obtained from (7a) by letting $u_I \to u_n$. The boundary condition at $x = \pm L/2$ is

$$u = 0. \tag{7c}$$

Since the nonlinear terms in (7a) and (7b) are very small in comparison to the linear terms, see (2b), a perturbation solution is sought in the form

$$u = u_0+\alpha u_1+\alpha^2 u_2+\dots \tag{8}$$

where $\alpha = (E_1/E_2)^3$. Substitution of (8) into (7) gives simple linear differential equations to be solved for each half cycle. An approximate solution, consisting of the first two terms in (8), is obtained for $n$ half cycles by the successive solutions of (7). At the end of the $n$th half cycle where $\Delta p < 0$

$$u_n = \frac{\delta A_n}{(1-4\lambda^2/\pi^2)}\cos\frac{\pi x}{L} \tag{9}$$

and

$$A_n = \left[1+\frac{n}{2}\left(\frac{E}{E_2}\right)^3 \frac{12\lambda^2\sigma_y}{(\pi^2/4\lambda^2-1)(1+E/E_1)E}\left(\frac{P}{AE}-\frac{\sigma_y}{E}\right)\right] > 0$$

$$\lambda^2 = \left(\frac{L}{2h}\right)^2\left(\frac{P}{A}\right)\left(\frac{1}{E}+\frac{1}{E_1}\right) > 0.$$

Consideration of (9) indicates that the maximum deflection at the end of the half cycle occurs at $x = 0$ and is given by

$$U_n = \frac{\delta A_n}{(1-4\lambda^2/\pi^2)}. \tag{10}$$

In addition, one can show that the deflection at each cross-section monotonically increases as the load increases. Thus, $U_n$ is maximum deflection occurring during the half cycle.

Now, the range of displacements for which the assumed deformation and (9) are valid will be determined. To establish the conditions for which (6) is satisfied, consider the differential form of (4c)

$$\mathrm{d}\sigma^+ = \frac{\mathrm{d}p}{A}+\mathrm{d}\left(\frac{pu}{Ah}\right); \qquad \mathrm{d}\sigma = \frac{\mathrm{d}p}{A}-d\left(\frac{pu}{Ah}\right). \tag{11}$$

As one might expect $-\mathrm{d}(pu/Ah)$ can be shown to increase monotonically with the load at each cross-section. Its maximum value occurs for $x = 0$ and $p = -P$. Thus, the maximum change in stress during the half cycle is the value of $\mathrm{d}\sigma$ for $x = 0$ and $p = -P$. The value is

$$\frac{\mathrm{d}p}{A}\left[1-\frac{U_n}{h(1-4\lambda^2/\pi^2)}\right]. \tag{12a}$$

By requiring that

$$\frac{U_n}{h} \leq 1-\left(\frac{2\lambda}{\pi}\right)^2 \tag{12b}$$

then the maximum change in stress is never positive and (6) is satisfied.

For loading satisfying (12b) both $\sigma^+$ and $\sigma^-$ decrease monotonically with load, and $\sigma^- \geq \sigma^+$ at each cross-section by (4c). Further consideration of (4c) indicates that the maximum stress as well as the maximum deflection occurs at $x = 0$ and $p = -P$. Thus, plastic deformation will occur in each element at every cross-section if for $x = 0$ and $p = -P$

$$\sigma^+ \leq -2\sigma_y. \tag{13a}$$

By using (4c), the above condition in terms of displacements is found to be

$$0 \leq \frac{U_n}{h} \leq 1-\frac{2\sigma_y A}{P}. \tag{13b}$$

When (13b) and (12b) are satisfied then the assumed deformation occurs and (9) is valid.

The deformation defined by (9) indicates that as $n$ increases $U_n$ becomes unbounded. Therefore, even a very small initial imperfection can develop into a large permanent deflection. In fact, conditions (12b) and (13b) indicate that the assumptions leading to the deflection given by (9) allow $U_n/h$ to change by as much as unity depending upon the magnitude of $\lambda$. Even a change in $U_n/h$ of one-half is large and probably critical for a column which is plastically designed on the basis of being nearly straight.

To obtain solutions for $u$ larger than those which satisfy (12b) and (13b), an analysis would have to be made for situations where only part of the elements are deformed into the plastic range. While the solution for these situations would be more tedious than that above, the results are now more obvious. As cycling continues, the top fiber can be expected to develop larger compressive stresses and thus accumulates contractive strain at increasing rates. The bottom fiber can be expected to develop regions of contractive elastic strains near $x = 0$, which later become regions of extensional elastic or plastic strains, and near $x = \pm L/2$ regions of smaller contractive plastic strains. Thus the bottom fiber will accumulate contractive strain at decreasing rate as cycling continues. The net result is that the curvature of the beam continues to increase which leads ultimately to collapse.

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(*Received* 7 *June* 1969)

**Абстракт**—Исследуются следствия существования пластических деформаций второго роди в применении к задачам периодической нагрузки колонн. Указывается на существование возможности для колонн, расчитаных с учетом коэффициента безопасности для низкого числа периодического нагружения, что эти колонны терают устойчивость для длительных сроков периодической нагрузки. Исследуются два примера. Первый это периодическое изменение между стационарными перемещениями концов, и второй—периодическое изменение между стационарной нагрузкой. В обоих примерах может появиться продольный изгиб, несмотря на то, что в первом примере должна быть принята правильна интерпретация сил связи при возможной потере устойчивости. Приводится элементарная пластическая модель, вызывающия эффектт второго рода, с целью использования ее во втором примере.